Nonhomogeneous Equations - Reduction of Order

The solution of a nonhomogeneous second-order linear equation

\[ y'' + p(x)y' + q(x)y = f(x) \]

is related to the solution of the corresponding homogeneous equation

\[ y'' + p(x)y' + q(x)y = 0. \]

Suppose \( y_1 \) is a particular solution to the homogeneous equation. Reduction of order “bootstraps” up from this particular solution to the general solution to the original equation.

The idea is to “guess” a general solution of the form \( y = vy_1 \), where \( v \) is to be determined. The derivatives are

\[ y' = v'y_1 + vy_1', \quad y'' = v''y_1 + 2v'y_1' + vy_1''. \]

Therefore,

\[
\begin{align*}
\quad y'' + p(x)y' + q(x)y &= v''y_1 + 2v'y_1' + vy_1'' + pv'y_1' + pv'y_1' + qvy_1 \\
&= v''y_1 + (2y_1' + py_1')v' + (y_1'' + p y_1' + q y_1)v \\
&= v''y_1 + (2y_1' + py_1')v'.
\end{align*}
\]

(I used the fact that \( y_1'' + py_1' + qy_1 = 0 \).)

Thus,

\[ v''y_1 + (2y_1' + py_1')v' = f(x). \]

Now set \( u = v' \), so \( u' = v'' \). The equation becomes

\[ u'y_1 + (2y_1' + py_1)u = f(x), \]

which is first order linear.

Now all you need to do is to solve the equation for \( u \), find \( v \), and finally obtain the general solution \( y \).

Rather than memorizing the last equation, it is better to understand the method, and to carry out the computation in each case.

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**Example.** First, I’ll check that this works in a situation where I know the general solution. Consider the equation

\[ y'' - y = 0. \]

The solutions are \( e^x \) and \( e^{-x} \). I’ll use \( e^x \) as \( y_1 \), so \( y = ve^x \). Then

\[ y' = v'e^x + ve^x, \quad y'' = v''e^x + 2v'e^x + ve^x. \]

Substitute:

\[
\begin{align*}
y'' - y &= v''e^x + 2v'e^x + ve^x - ve^x \\
&= v''e^x + 2v'e^x.
\end{align*}
\]

Now I have

\[ v''e^x + 2v'e^x = 0. \]
Therefore, \( v'' + 2v' = 0 \).

Let \( u = v' \), so \( u' = v'' \):
\[ u' + 2u = 0. \]

This is \( u' = -2u \), and exponential decay equation. The solution is
\[ u = ce^{-2x}. \]

Since \( u = v' \), I have
\[ v' = ce^{-2x}. \]

Integrate:
\[ v = -\frac{c}{2}e^{-2x} + c_2 = c_1 e^{-2x} + c_2. \]

Finally, put \( y = ve^x \), so
\[ y = c_1 e^{-x} + c_2 e^x. \]

This agrees with the solution you'd obtain using the characteristic equation technique.

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**Example.** Solve \((D^2 + 1)y = \sec x\).

The corresponding homogeneous equation is
\[ (D^2 + 1)y = 0. \]

By looking at the characteristic equation, you can see that \( \cos x \) and \( \sin x \) are solutions.

Let \( y = v\cos x \), so
\[ y' = v'\cos x - v\sin x, \quad y'' = v''\cos x - 2v'\sin x - v\cos x. \]

Substitute:
\[ y'' + y = v''\cos x - 2v'\sin x - v\cos x + v\cos x = v''\cos x - 2v'\sin x. \]

The equation is
\[ v''\cos x - 2v'\sin x = \sec x. \]

Let \( u = v' \), so \( u' = v'' \):
\[ u'\cos x - 2u\sin x = \sec x, \]
\[ u' - 2\frac{\sin x}{\cos x}y = (\sec x)^2. \]

This is first order linear; the integrating factor is
\[ l = \exp \int -2\frac{\sin x}{\cos x} \, dx = \exp 2\ln \cos x = (\cos x)^2. \]

Hence,
\[ u(\cos x)^2 = \int (\cos x)^2 \, dx = x + C, \]
\[ u = x(\sec x)^2 + C(\sec x)^2. \]

Let \( u = v' \), so
\[ v' = x(\sec x)^2 + C(\sec x)^2, \]
\[ v = x(\sec x)^2 + C(\sec x)^2, \]
\[ v = x \tan x - \ln |\sec x| + C \tan x + D. \]

Here is the work for the integral:

\[
\begin{align*}
\frac{d}{dx} &+ \int dx \\
+ &x \quad (\sec x)^2 \\
- &1 \quad \tan x \\
+ &0 \quad \ln |\sec x| \\
\int x(\sec x)^2 \, dx &= x \tan x - \ln |\sec x| + C.
\end{align*}
\]

Finally, \( y = v \cos x \), so

\[ y = v \cos x = x \sin x - \cos x \ln |\sec x| + C \sin x + D \cos x. \]

**Example.** Solve \( xy'' + 2y' + xy = 0 \), if \( y_1 = \frac{\sin x}{x} \) is a homogeneous solution.

Let \( y = \frac{\sin x}{x} \), so

\[ y' = v' \frac{\sin x}{x} + v \frac{\cos x}{x} - \frac{\sin x}{x^3}, \]

\[ y'' = v'' \frac{\sin x}{x} + 2v' \left( \frac{\cos x}{x} - \frac{\sin x}{x^2} \right) - v \left( \frac{\sin x}{x} + 2 \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} \right). \]

Plug this awful stuff into the original equation and simplify:

\[
\begin{align*}
xy'' + 2y' + xy &= v'' \sin x + 2v' \left( \frac{\cos x}{x} - \frac{\sin x}{x} \right) - v \left( \frac{\sin x}{x} + 2 \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} \right) \\
+ &2v' \frac{\sin x}{x} + v \left( \frac{\sin x}{x} + 2 \frac{\cos x}{x^2} - 2 \frac{\sin x}{x^3} \right) \\
&= v'' \sin x + 2v' \cos x.
\end{align*}
\]

Therefore,

\[ v'' \sin x + 2v' \cos x = 0. \]

Let \( u = v' \), so \( u' = v'' \):

\[ u' \sin x + 2u \cos x = 0. \]

Separate variables:

\[
\int \frac{du}{u} = -2 \int \frac{\cos x}{\sin x} \, dx,
\]

\[ \ln |u| = -2 \ln |\sin x| + C, \]

\[ u = C_3 (\csc x)^3. \]

Now \( u = v' \), so

\[ v' = C_3 (\csc x)^3, \]

\[ v = -C_3 \cot x + C_2. \]
Finally, \( y = \frac{\sin x}{x} \), so
\[
y = \frac{\sin x}{x} = C_1 \cos x \frac{x}{x} + C_2 \sin x \frac{x}{x}.
\]

Here’s a way to save writing and simplify the computation. The idea is to carry through the computation with \( y = vy_1 \) and substitute for \( y_1 \) after simplifying, rather than plugging in \( y_1 \) at the start.

To do this, write \( y = vy_1 \), so
\[
y' = v' y_1 + vy_1' \quad \text{and} \quad y'' = v'' y_1 + 2v' y_1' + vy_1'.'
\]
Substitute:
\[
x y'' + 2y' + xy = xv'' y_1 + 2xv' y_1' + xvy_1'' + 2xy_1' + xvy_1' = xv'' y_1 + 2v'(xy_1' + y_1) + v(xy_1'' + 2y_1' + xy_1)
\]
The last term dies, because \( y_1 \) was a solution. I’m left with
\[
xv'' y_1 + 2v'(xy_1' + y_1) = 0.
\]
Now \( y_1 = \frac{\sin x}{x} \), so
\[
y_1' = \frac{\cos x}{x} - \frac{\sin x}{x^2}.
\]
Hence,
\[
xv'' \left( \frac{\sin x}{x} \right) + 2v' \left( x \left( \frac{\cos x}{x} - \frac{\sin x}{x^2} \right) + \frac{\sin x}{x} \right) = 0.
\]
Simplifying, I get
\[
v'' \sin x + 2v' \cos x = 0.
\]
After this, the computation continues as before. I’ve saved some writing, and I’ve also avoided having to compute \( y_1'' \).  

\( \square \)