Duality in Linear Programming

For every LPP there is a related unique LPP involving the same data that also describes the original problem. The given original problem is called a primal problem. This can be solved by transposing the rows and columns of the algebraic statements of the problem. Inverting the problem in this way results in a dual problem. Duality is an extremely important concept in LPP. The various useful aspects of this are:

1. If the primal problem contains a large number of constraints and a small number of variables, the computational procedure can be reduced by converting it into dual and then solve it.
2. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem. This is termed as post optimality or sensitivity analysis.
3. Calculation of the dual checks the accuracy of the primal solution.

Dual problem when the primal is in the canonical form.

The general LPP in canonical form is written as

Maximize \( z = c_1x_1 + c_2x_2 + \ldots + c_nx_n, \)

Subject to

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq b_1, \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq b_2, \quad \ldots \ldots (1) \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq b_m, \\
  x_1, x_2, \ldots, x_n & \geq 0.
\end{align*}
\]

If the above is referred as primal, then its associated dual will be

Minimize \( w = b_1y_1 + b_2y_2 + \ldots + b_my_m, \)

Subject to

\[
\begin{align*}
  a_{11}y_1 + a_{12}y_2 + \ldots + a_{m1}y_m & \geq c_1, \\
  a_{21}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m & \geq c_2, \quad \ldots \ldots (2) \\
  a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{mn}y_m & \geq c_n, \\
  y_1, y_2, \ldots, y_m & \geq 0.
\end{align*}
\]

where the associated dual variables \( y_1, y_2, \ldots, y_m \geq 0. \)

Using the matrix notation, the primal can be written as

Maximise \( z = c^T x, \) where \( c^T \in \mathbb{R}^n, x \in \mathbb{R}^n. \)

Subject to

\[
\begin{align*}
  Ax & \leq b, \\
  x & \geq 0 \quad \text{where } A \text{ is an } m \times n \text{ matrix and } b^T \in \mathbb{R}^m
\end{align*}
\]
The dual problem can be written as

Minimise \( y = b^T w \)

Subject to

\[ A^T w \geq c, \]
\[ w \geq 0. \]

Note

1. If the primal contains \( n \) variables and \( m \) constraints, then the dual will contain \( m \) variables and \( n \) constraints.
2. The maximization in the primal becomes the minimization in the dual and vice versa.
3. The maximization problem has (\( \leq \)) constraints while the minimization problem has (\( \geq \)) constraints.
4. The constants \( c_1, c_2, \ldots, c_m \) in the objective function of the primal appear in the constraints in the dual.
5. The constants \( b_1, b_2, \ldots, b_m \) in the constraints of the primal appear in the objective function of the dual.
6. The variables in both problems are non-negative.

**Problem 1**

Construct the dual of the problem

Minimize \( z = 3x_1 - 2x_2 + 4x_3 \),

Subject to constraints

\[
\begin{align*}
3x_1 + 5x_2 + 4x_3 & \geq 7 \\
6x_1 + x_2 + 3x_3 & \geq 4 \\
7x_1 - 2x_2 - x_3 & \leq 10 \\
x_1 - 2x_2 + 5x_3 & \geq 3 \\
4x_1 + 7x_2 - 2x_3 & \geq 2 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Remark

1. If the primal problem is in standard form, then the dual variables will always be unrestricted in sign (irrespective of the sign of the primal ones). This condition however will be satisfied only for those primal constraints that were initially in equation.
2. If some primal variables are unrestricted in sign, then those dual constraints will be equations that correspond to the said primal variables.
Problem 2

Construct the dual of

Maximize \[ z = 3x_1 + 10x_2 + 2x_3, \]

Subject to

\[ \begin{align*}
2x_1 + 3x_2 + 2x_3 & \leq 3 \\
3x_1 - 2x_2 + 4x_3 & = 3 \\
x_1, x_2, x_3 & \geq 0.
\end{align*} \]

Problem 3

Obtain the dual of

Maximize \[ z = x_1 + 3x_2 - 2x_3 + 5x_4, \]

Subject to

\[ \begin{align*}
3x_1 - x_2 + x_3 - 4x_4 & = 2 \\
5x_1 + 3x_2 - x_3 - 2x_4 & = 3 \\
x_1, x_2 & \geq 0 \\
\text{and } x_3, x_4 & \text{ are unrestricted in sign.}
\end{align*} \]

Properties of Primal and Dual optimal solutions.

The final simplex table giving optimal solution to the primal problem also contains the optimal solution to the dual and vice versa.

1. The optimal value of the objective function of the primal is equal to the optimal value of the objective function of the dual.

\[ Z_{\text{max.}} = W_{\text{min.}} \text{ or } Z_{\text{min.}} = W_{\text{max.}} \]

2. If the primal (dual) variable corresponding to a slack starting variable in the dual (primal) problem, its optimum value is directly read off from the net evaluation row of the optimum dual (primal) simplex table, as the net evaluation corresponding to this slack variable.

3. If the primal (dual) variable corresponds to an artificial starting variable in the dual (primal) problem, its optimum value is directly read off from the net evaluation row of the optimum dual (primal) simplex table, as the net evaluation corresponding to this artificial variable after deleting the constant \( M \).
Problem

Using duality solve the following LPP

Maximize \[ z = 3x_1 - 2x_2, \]

Subject to

\[ \begin{align*}
x_1 & \leq 4 \\
x_2 & \leq 6 \\
x_1 + x_2 & \leq 5 \\
x_1 & \leq -1 \\
x_1, x_2 & \geq 0.
\end{align*} \]

Dual Simplex Algorithm

1. Convert the minimization LPP into that of maximization, if it is in the minimization form. Convert the \( \geq \) type in equations, representing the constraints of the given LPP, if any, into \( \leq \) type.
2. Introduce the slack variables in the constraints of the given problem and obtain an initial basic solution. Put this solution in the starting dual simplex table.
3. Test the nature of \( z_j - c_j \) in the simplex table.
   (i) If all \( z_j - c_j \) and \( x_{Bi} \) are non-negative, for all \( i \) and \( j \), then an optimum basic feasible solution has been obtained.
   (ii) If all \( z_j - c_j \) are non-negative and at least one of \( x_{Bi} < 0 \), then go to step 4.
   (iii) If at least one \( z_j - c_j < 0 \), the method is not applicable to the given problem.
4. Select the most negative \( x_{Bi} \). The corresponding basis vector then leaves the basis \( Y_B \).
   Let \( x_{Bk} \) be the most negative basic variable so that \( y_k \) leaves the basis \( Y_B \).
5. Test the nature of \( y_{kj}, j = 1, 2, \ldots, n \).
   (i) If all \( y_{kj} \geq 0 \), there does not exist any feasible solution to the given problem.
   (ii) If at least one \( y_{kj} < 0 \), compute the replacement ratio \( \{(z_j - c_j) / y_{kj}, \ for \ y_{kj} < 0\} \), \( j = 1, 2, \ldots, n \). and choose the maximum of these. The corresponding column vector, say, \( y_r \), then enters the basis \( Y_B \).
6. Test the new iterated dual simplex table for optimality.

Repeat the procedure until either an optimum feasible solution has been obtained (in a finite number of steps) or there is an indication of the non-existence of a feasible solution.

Problem

Use dual simplex method to

Minimize \[ z = x_1 + 2x_2 + 3x_3 \]

Subject to

\[ \begin{align*}
x_1 - x_2 + x_3 & \geq 4 \\
x_1 + x_2 + 2x_3 & \leq 8 \\
x_2 - x_3 & \geq 2 \\
x_1, x_2, x_3 & \geq 0.
\end{align*} \]


**Economic Interpretations**

The *shadow/dual price* equivalency states that the optimal value of the $i^{th}$ dual variable is the amount by which the optimal value of the primal objective function will change per allowable unit increase in $b_i$, all other parameters held as constant. For a maximization primal where the $\leq$ constraints represent limited resources, the corresponding dual price is measure of the marginal value of additional units of a resource.

Base on these economic interpretations, the canonical dual problem is often called a *pricing problem*. Its optimal solution yields the minimum prices that management should accept for selling or liquidating resources; the optimal value of the objective function is the minimum acceptable payment for all these resources.

One of the important properties of the duality theory is, for any pair of feasible primal and dual solutions, \( \sum_{i=1}^{n} c_i x_i \leq \sum_{i=1}^{m} b_i y_i = w \). At the optimum, the relationship holds as a strict equation.

By this property, we have

\[
z = \sum_{j=1}^{m} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = w
\]

The strict equality holds when both primal and dual solutions are *optimal*. The primal problem represents a resource allocation model and $z$ as profit dollars.

The dual variables $y_i$'s represents the worth per unit of resource $i$ and if $z < w$. That is, (profit) < (worth of resources), the corresponding primal and dual solutions cannot be optimal.

**Fundamental theorem of Duality**

If both the primal and dual problems have feasible solutions, then both have optimal solutions and $\max z = \min w$.

**Remark**

The dual of the dual is the primal problem.